数理解析研究所講究録
第 815 巻 1992 年 123-133

MAT and strong C^0-equivalence

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In this talk, we describe a method of drawing a picture of the zero locus of a polynomial-germ $f : (\mathbf{R}^3, 0) \to (\mathbf{R}, 0)$, and make several claims on phase of the germ f. Using our method, we can draw a picture of the Briançon-Speder's family in [2], Oka's family in [11], and so on, and make an elementary explanation that these families do not strongly C^0-trivial in the S.Koike's sense in [8]. Moreover we construct a family which admits a MAT via some modification, but not strongly C^0-trivial. As a consequence we give a counterexample to a conjecture stated in T.-C.Kuo [9].

The talk will proceed in the following way. In §1, we review and modify several facts in the theory of toric varieties and toric modification, mainly due to V.I.Danilov[3,4], and T.Oda[10]. In §2, we give the definition of strong C^0-equivalence, MAT, and their generalization. We also discuss some elementary facts on these equivalences. In §3, we describe a way to draw a picture of the Briançon-Speder family, and so on, and give an elementary explanation why they are or are not strongly C^0-trivial.

The author would like to express his hearty thanks to S.Koike. Several discussions with him were helpful for preparing this article.

1. Toric varieties and modifications. We recall and modify here the construction of the toric variety P_Δ associated with a polyhedron Δ, mainly due to V.I.Danilov [3,4] and T.Oda [10].

Set $\mathbf{R}_+ = \{x \in \mathbf{R} | x \geq 0\}$. Let Δ be a convex polyhedron in $\mathbf{R}^n$ whose faces are defined by linear equations and linear inequalities with rational coefficients. We denote $F < \Delta$, if F is a face of Δ. With each face F of Δ we associate a cone σ_F in $\mathbf{R}^n$: to do this we take a point $m \in \mathbf{R}^n$ lying inside the face F, and we set

$$\sigma_F = \mathrm{Cone}(\Delta, F) = \bigcup_{r \geq 0} r \cdot (\Delta - m).$$

The system $\{\sigma_F^\vee\}$, as F ranges over the faces of Δ, is a *fan*, which we denote by Σ_Δ. With each face F of Δ, we denote R_F the semi-group ring generated by the semi-group $\sigma_F \cap \mathbf{Z}^n$ over the real field $\mathbf{R}$ and set $U_F = \mathrm{Spec}(R_F)$. We denote $U_F(\mathbf{R})$ the set of real points of the affine scheme U_F. In other words, $U_F(\mathbf{R})$ is the set of unitary semi-group homomorphisms from $\sigma_F \cap \mathbf{Z}^n$ to $\mathbf{R}$. Let $m_1, ..., m_p$ be generators of $\sigma_F \cap \mathbf{Z}^n$

as a semi-group. Then there is an injection of $U_F(\mathbf{R})$ to $\mathbf{R}^p$ defined by $u \mapsto (u(m_1), ..., u(m_p))$. The image of this map has a structure of real algebraic varieties. Let $U_F(\mathbf{R}_+)$ be the set of semi-group homomorphisms from $\sigma_F \cap \mathbf{Z}^n$ to $\mathbf{R}_+$. The image of $U_F(\mathbf{R}_+)$ is a semi-algebraic subset, and is homeomorphic to σ_F. The real spectrum $\mathcal{R}$-Spec(R_F) is naturally homeomorphic to the ultrafilter completion $\widehat{U_F(\mathbf{R})}$ of $U_F(\mathbf{R})$ in the lattice of all semi-algebraic subsets of $U_F(\mathbf{R})$. (See [1].)

If F_1 is a face of F, then $\sigma_F^\vee$ is a face of $\sigma_{F_1}^\vee$, thus U_{F_1} (resp. $U_{F_1}(\mathbf{R})$, $U_{F_1}(\mathbf{R}_+)$) is identified with an open subset of U_F (resp. $U_F(\mathbf{R})$, $U_F(\mathbf{R}_+)$), and $\mathcal{R}$-Spec(R_{F_1}) can be identified with part of $\mathcal{R}$-Spec(R_F). These identifications allow us to glue together of U_F, $U_F(\mathbf{R})$, $U_F(\mathbf{R}_+)$, and $\mathcal{R}$-Spec(R_F), as F ranges over the faces of Δ, which are denoted by P_Δ, $P_\Delta(\mathbf{R})$, $P_\Delta(\mathbf{R}_+)$, and $\widehat{P_\Delta(\mathbf{R})}$ respectively. Let P be a vertex of Δ. A polyhedron Δ is *regular at* P if the Cone(Δ, P) is generated by a basis of $\mathbf{Z}^n$. A polyhedron is *regular* if it is regular at all vertices. If Δ is regular, then $P_\Delta(\mathbf{R})$ is a non-singular variety. We have that $P_\Delta(\mathbf{R}_+)$ is homeomorphic to Δ. To each face F of Δ, there is an associated closed subset in P_Δ (resp. $P_\Delta(\mathbf{R})$, $P_\Delta(\mathbf{R}_+)$, $\widehat{P_\Delta(\mathbf{R})}$), which is canonically isomorphic to P_F (resp. $P_F(\mathbf{R})$, $P_F(\mathbf{R}_+)$, $\widehat{P_F(\mathbf{R})}$). We allow a certain freedom in the notation and denote it by the same symbol P_F (resp. $P_F(\mathbf{R})$, $P_F(\mathbf{R}_+)$, $\widehat{P_F(\mathbf{R})}$). If F is a face of Δ, then $P_F \subset P_\Delta$ (resp. $P_F(\mathbf{R}) \subset P_\Delta(\mathbf{R})$, $P_F(\mathbf{R}_+) \subset P_\Delta(\mathbf{R}_+)$, $\widehat{P_F(\mathbf{R})} \subset \widehat{P_\Delta(\mathbf{R})}$). Set theoretically, $P_F \cap P_{F'} = P_{F\cap F'}$, $P_F(\mathbf{R}) \cap P_{F'}(\mathbf{R}) = P_{F\cap F'}(\mathbf{R})$, and $P_F(\mathbf{R}_+) \cap P_{F'}(\mathbf{R}_+) = P_{F\cap F'}(\mathbf{R}_+)$. Let $T_F = P_F(\mathbf{R}) - \bigcup_{G<F} P_G(\mathbf{R})$. Then $P_\Delta(\mathbf{R}) = \coprod_{F<\Delta} T_F$. (The canonical stratification of $P_\Delta(\mathbf{R})$.)

Let Δ_1, Δ_2 be polyhedra in $\mathbf{R}^n$. We say Δ_1 *majorizes* Δ_2 if there exists an order preserving map φ from faces of Δ_1 to faces of Δ_2 such that Cone$(\Delta_2, \varphi(F)) \subset$ Cone(Δ_1, F) for any face F of Δ_1.

If Δ_1 majorizes Δ_2, then there are canonical maps $P_{\Delta_1} \to P_{\Delta_2}$, $P_{\Delta_1}(\mathbf{R}) \to P_{\Delta_2}(\mathbf{R})$, $P_{\Delta_1}(\mathbf{R}_+) \to P_{\Delta_2}(\mathbf{R}_+)$, and $\widehat{P_{\Delta_1}(\mathbf{R})} \to \widehat{P_{\Delta_2}(\mathbf{R})}$, induced by the natural embedding of semi-group rings

$$\mathbf{R}[\mathrm{Cone}(\Delta_2, \varphi(F)) \cap \mathbf{Z}^n] \to \mathbf{R}[\mathrm{Cone}(\Delta_1, F) \cap \mathbf{Z}^n].$$

EXAMPLE 1. Let Δ_1 be a parallelogram $A_1A_2B_1B_2$ so that the segment A_1B_1 is parallel to A_2B_2. Let Δ_2 be a segment AB. Then Δ_1 majorizes Δ_2 by the map defined by $A_i \mapsto A, B_i \mapsto B, i = 1, 2$. This gives a $\mathbf{R}P^1$-bundle $P_{\Delta_1}(\mathbf{R}) \to P_{\Delta_2}(\mathbf{R}) = \mathbf{R}P^1$.

EXAMPLE 2. Let Δ be a convex polyhedron in $\mathbf{R}^n$ coinciding with $\mathbf{R}^n_+$ outside some compact set. Then Δ majorizes $\mathbf{R}^n_+$ and we get maps $\rho_\Delta : P_\Delta(\mathbf{R}) \to P_{\mathbf{R}^n_+}(\mathbf{R}) = \mathbf{R}^n$, $\rho_{\Delta,+} : P_\Delta(\mathbf{R}_+) \to P_{\mathbf{R}^n_+}(\mathbf{R}_+) = \mathbf{R}^n_+$.

We call ρ_Δ a *real toric modification* of $\mathbf{R}^n$ defined by Δ. In fact, ρ is proper and is an isomorphism over $\mathbf{R}^n - \{0\}$. The exceptional set $\rho^{-1}(0)$ consists of the varieties P_F, where F ranges over the compact faces of Δ.

NOTATIONS. Using the same notation in example 2, we set $m = 1 + \sum_{i=1}^n e_i 2^{i-1}$, for $e_i \in \{0,1\}$. Let us put $A_m = \{(x_1, ..., x_n) \in \mathbf{R}^n | \text{sign} x_i = (-1)^{e_i}\}$, and $A_m(\Delta) =$ closure of $\rho_\Delta^{-1}(A_m)$ in $P_\Delta(\mathbf{R})$. Each $A_m(\Delta)$ is homeomorphic to Δ, and $P_\Delta(\mathbf{R}) = \bigcup_{1 \le m \le 2^n} A_m(\Delta)$. Set $A(\Delta)$ the set obtained by gluing of $A_m(\Delta)$ along non-compact faces of Δ, and $\tilde{\rho}_\Delta$ the natural map of $A(\Delta)$ to $\mathbf{R}^n$.

Let $f(x)$ be a real analytic function of n variables $x = (x_1, ..., x_n)$ in a neighbourhood of the origin of $\mathbf{R}^n$, and

$$\sum_\nu c_\nu x_1^{\nu_1} \cdots x_n^{\nu_n},\ \nu = (\nu_1, ..., \nu_n)$$

be the Taylor expansion of $f(x)$ at the origin. Let $\Gamma_+(f)$ be the convex hull in $\mathbf{R}^n$ of the set

$$\{\nu + \mathbf{R}_+^n | c_\nu \neq 0\}.$$

Let Δ be a regular polyhedron in $\mathbf{R}^n$ coinciding with $\mathbf{R}_+^n$ outside some compact set. For a face F of Δ, there are $(n-1)$-dimensional faces $F_1, ..., F_s$ of Δ such that $F = \bigcap_j F_j$. Set a^j be a primitive vector normal to F_j, and set $\ell_j = \min\{\langle a^j, \nu\rangle | \nu \in \Gamma_+(f)\}$. Define the set $\gamma = \gamma(F)$ by $\Gamma_+(f) \cap \{\nu | \langle a^j, \nu \rangle = \ell_j, j = 1, ..., s\}$, and set $f_\gamma = \sum_{\nu \in \gamma} c_\nu x_1^{\nu_1} \cdots x_n^{\nu_n}$.

Let $Z = Z_\Delta(f)$ (resp. $Z = \tilde{Z}_\Delta(f)$) be the proper transform of $f^{-1}(0)$ via ρ_Δ (resp. $\tilde{\rho}_\Delta$). Then we have the following lemmas.

LEMMA. $Z_\Delta(f) \cap T_F \cong E_\gamma(f) \times (\mathbf{R} - \{0\})^{\dim F - \dim \gamma}$, where $E_\gamma(f)$ is the algebraic set defined by $f_\gamma = 0$ in T_γ.

LEMMA. The following statements are equivalent.

1) Z intersects transversely with T_F.

2) $(\partial f_\gamma / \partial x_1, ..., \partial f_\gamma / \partial x_n)$ is not zero except $\{x_1 \cdot \cdots \cdot x_n = 0\}$.

We say $f(x)$ is *non-degenerate* if $(\partial f_\gamma / \partial x_1, ..., \partial f_\gamma / \partial x_n)$ is not zero except $\{x_1 \cdot \cdots \cdot x_n = 0\}$ for any compact face γ of $\Gamma_+(f)$.

2. Definitions. Let $F(x,t) = f_t(x)$ be a real analytic family of real analytic functions of n-variables $x = (x_1, ..., x_n)$ parametrized by $t = (t_1, ..., t_m) \in I$, where I is a compact cube $[a_1, b_1] \times ... \times [a_m, b_m]$. For the sake of notational simplicity, we do not distinguish germs and their representatives. Let $\pi : (X, E) \to (\mathbf{R}^n, 0)$ be a proper analytic modification.

DEFINITION. We say that f_t admits a *modified analytic trivialization (MAT) via π along I* if there exists t-level preserving analytic isomorphism $H : (X, E) \times I \to (X, E) \times I$ such that $F \circ (\pi \times \mathrm{id}_I) \circ H$ is independent of t, and that H induces a t-level preserving homeomorphism h of $(\mathbf{R}^n, 0) \times I$.

$$\begin{tikzcd}
(X,E) \times I \arrow[r, "H"] \arrow[d, "\pi \times \mathrm{id}_I"'] & (X,E) \times I \arrow[d, "\pi \times \mathrm{id}_I"] \\
(\mathbf{R}^n, 0) \times I \arrow[r, "h"] & (\mathbf{R}^n, 0) \times I
\end{tikzcd}$$

DEFINITION. We say that f_t is *strongly C^0-trivial along I* if there exists a t-level preserving homeomorphism $h(x,t) = (h_t(x), k(t))$ of $(\mathbf{R}^n, 0) \times I$ such that $F \circ h$ is independent on t and that h satisfies the following conditions.
1) For any analytic germ $\alpha : ([0,\varepsilon), 0) \to (\mathbf{R}^n, 0) \times \{t_0\}$ with $f_{t_0} \circ \alpha \equiv 0$, $h_t \circ \alpha$ is analytic.
2) For any analytic germs $\alpha, \beta : ([0,\varepsilon), 0) \to (\mathbf{R}^n, 0) \times \{t_0\}$ with $f_{t_0} \circ \alpha \equiv 0$, $f_{t_0} \circ \beta \equiv 0$, α and β have a same tangent if and only if $h_t \circ \alpha$ and $h_t \circ \beta$ have.

The definition of strong C^0-equivalence is due to S.Koike [8].

DEFINITION. We say that f_t is *tangentially C^0-trivial along I* if there exists a t-level preserving homeomorphism $h(x,t) = (h_t(x), k(t))$ of $(\mathbf{R}^n, 0) \times I$ such that $F \circ h$ is independent on t and that h satisfies the following conditions.
1) For any germ $\alpha : ([0,\varepsilon), 0) \to (\mathbf{R}^n, 0) \times \{t_0\}$, the tangent direction of $h_t \circ \alpha$ at 0 can be defined, if the tangent direction of α at 0 can be.
2) For any analytic germs $\alpha, \beta : ([0,\varepsilon), 0) \to (\mathbf{R}^n, 0) \times \{t_0\}$, α and β have a same tangent if and only if $h_t \circ \alpha$ and $h_t \circ \beta$ have.

DEFINITION ([7] p.221). Let $n \geq 2$ and S be the unit sphere with center at the origin in $\mathbf{R}^n$. Let $\pi_1 : \mathbf{R} \times S \to \mathbf{R}^n$ by $(t,v) \mapsto tv$. This is a degree 2 proper map of real analytic manifolds, which is called *double oriented blowing up* of $\mathbf{R}^n$. The map π_1 induces $\pi_2 : \overline{X} = \mathbf{R}_+ \times S \to \mathbf{R}^n$, which is called *(simple) oriented blowing up*. It also induces $\pi_3 : X = \mathbf{R} \times S/\mathbf{Z}_2 \to \mathbf{R}^n$, where $\mathbf{Z}_2 = \mathbf{Z}/2\mathbf{Z} = \{\pm 1\}$ acts on $\mathbf{R} \times S$ by $(t,v) \mapsto (-t,-v)$. This π_3 is called the *(non-oriented) blowing up* of $\mathbf{R}^n$ with center $0 \in \mathbf{R}^n$.

LEMMA. Set $\Delta = \{(\nu_1, ..., \nu_n) \in \mathbf{R}^n_+ | \nu_1 + ... + \nu_n \geq 1\}$. Then ρ_Δ (resp. $\tilde{\rho}_\Delta$) is the blowing up (resp. oriented blowing up) of $\mathbf{R}^n$ with center $0 \in \mathbf{R}^n$.

LEMMA. f_t is tangentially C^0-trivial along I, if and only if, there exist a homeomorphisms $H : (\overline{X}, \pi_2^{-1}(0)) \times I \to (\overline{X}, \pi_2^{-1}(0)) \times I$, and a topological trivialization $h : (\mathbf{R}^n, 0) \times I \to (\mathbf{R}^n, 0) \times I$ of f_t, that satisfies the following commutative diagram.

$$\begin{array}{ccc}
(\overline{X}, \pi_2^{-1}(0)) \times I & \xrightarrow{H} & (\overline{X}, \pi_2^{-1}(0)) \times I \\
{\scriptstyle \pi_2 \times \mathrm{id}_I}\downarrow & & \downarrow{\scriptstyle \pi_2 \times \mathrm{id}_I} \\
(\mathbf{R}^n, 0) \times I & \xrightarrow[h]{} & (\mathbf{R}^n, 0) \times I
\end{array}$$

It seems to be hard to find a similar lemma for strong C^0-triviality. But we have that a strong C^0-trivialization induces a topological trivialization of proper transforms of $f_t^{-1}(0)$, $t \in I$, via the oriented blowing up π_2.

These suggest the following definitions.

DEFINITION. Let $r = 0, 1, 2, ..., \infty, \mathrm{or}, \omega$. Let $\pi : X \to (\mathbf{R}^n, 0)$ be an analytic map. A homeomorphism $h : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ (resp. a t-level preserving homeomorphism $h : (\mathbf{R}^n, 0) \times I \to (\mathbf{R}^n, 0) \times I$) is said to be *$C^r$-liftable via π* if there exists a C^r-isomorphism $H : X \to X$ (resp. a t-level preserving C^r-isomorphism $H : X \times I \to X \times I$) with $\pi \circ H = h \circ \pi$ (resp. $(\pi \times \mathrm{id}_I) \circ H = h \circ (\pi \times \mathrm{id}_I)$).

REMARK. Let $h : (\mathbf{R}^n, 0) \times I \to (\mathbf{R}^n, 0) \times I$ be a t-level preserving homeomorphism that topologically trivialize a family f_t.

1) If the trivialization h of a family f_t is C^0-liftable via the oriented blowing up, then h gives a tangential C^0-trivialization of f_t.

2) If the trivialization h of f_t is C^ω-liftable via π, then h gives a modified analytic trivialization of f_t via π.

We can generalize this property on lifting in the following form.

DEFINITION. Let $(\mathcal{D}, <)$ be a finite set with partial ordering with minimum element o. We associate a space X_α for $\alpha \in \mathcal{D}$, and a map $\pi_{\alpha\beta} : X_\beta \to X_\alpha$ for $\alpha, \beta \in \mathcal{D}$ with $\alpha < \beta$. Suppose that $X_o = (\mathbf{R}^n, 0)$ and $\pi_{\alpha\beta} \circ \pi_{\beta\gamma} = \pi_{\alpha\gamma}$, for $\alpha, \beta, \gamma \in \mathcal{D}$ with $\alpha < \beta < \gamma$. Let $\psi : \mathcal{D} \to \{0, 1, 2, ..., \infty, \omega\}$ be a map. A homeomorphism $h : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ (resp. a t-level preserving homeomorphism $h : (\mathbf{R}^n, 0) \times I \to (\mathbf{R}^n, 0) \times I$) is said to be *$\psi$-liftable via $\mathcal{D}$* if there exist $C^{\psi(\alpha)}$-isomorphisms $H_\alpha : X_\alpha \to X_\alpha$ (resp. a t-level preserving $C^{\psi(\alpha)}$-isomorphism $H_\alpha : X_\alpha \times I \to X_\alpha \times I$), for $\alpha \in \mathcal{D}$, such that $\pi_{\alpha\beta} \circ H_\beta = H_\alpha \circ \pi_{\alpha\beta}$ (resp. $(\pi_{\alpha\beta} \times \mathrm{id}_I) \circ H_\beta = H_\alpha \circ (\pi_{\alpha\beta} \times \mathrm{id}_I)$), for $\alpha, \beta \in \mathcal{D}$ with $\alpha < \beta$, and $H_o = h$.

Let $f : X \to Y$ be a map between two manifolds. A point p in X is said to be *topologically regular point* if f is topologically right-left equivalent to a regular map near p. A point p is said to be *topologically critical point* of f if p is not a topologically regular point of f.

3. Drawing a picture of example. An analysis on examples of polynomial-germs f with 3 variables will proceed in the following way.

Draw a picture of $E_\gamma(f)$ in $A_m(\Gamma_+(f))$, for compact faces γ of $\Gamma_+(f)$, and patch them together in $A(\Gamma_+(f))$. Find Δ with regular $Z_\Delta(f)$, if it exists, and compare $Z_\Delta \cap T_F$'s with $E_\gamma(f)$'s, for faces F of Δ. If we concern with a family f_t, then try to find a polyhedron Δ so that $Z_\Delta(f_t)$ are simultaneously smooth. If we concern with tangential C^0-triviality, choose Δ majorizing Δ_0, where $\Delta_0 = \{(\nu_1,\nu_2,\nu_3) \in \mathbf{R}^3_+ | \nu_1+\nu_2+\nu_3 \geq 1\}$. Then concentrate on the map $\tilde{Z}_\Delta(f_t) \to \tilde{Z}_{\Delta_0}(f_t)$. If we concern with construction of a counterexample to the conjecture stated in §2 in [9], choose Δ majorizing some Δ' (Δ_0, etc.), and concentrate on the map $Z_\Delta(f_t) \to Z_{\Delta'}(f_t)$.

3-1. Briançon-Speder's family in [2]. Let $f_t(x_1,x_2,x_3) = x_3^5 + tx_2^6x_3 + x_1x_2^7 + x_1^{15}$. Set $\Delta_1 = \{(\nu_1,\nu_2,\nu_3) \in \mathbf{R}^3_+ | \nu_1 + 2\nu_2 + 3\nu_3 \geq 6\}$, and $\Delta_2 = \{(\nu_1,\nu_2,\nu_3) \in \mathbf{R}^3_+ | \nu_1+\nu_2+\nu_3 \geq 8, \nu_1+2\nu_2+3\nu_3 \geq 12, 2\nu_1 + 2\nu_2 + 3\nu_3 \geq 18\}$. The polyhedron Δ_2 majorizes Δ_0, and Δ_1. Define Δ_3 by $\{(\nu_1,\nu_2,\nu_3) \in \Delta_2 | \nu_1+\nu_2+\nu_3 \geq 8+\varepsilon_1, \nu_1+\nu_2+2\nu_3 \geq 9+\varepsilon_2\}$, choosing small positive rational numbers ε_1 and ε_2. Then Δ_3 is a regular polyhedron majorizing Δ_2. Set $F_1 = \Delta_2 \cap \{\nu_1+2\nu_2+3\nu_3 = 12\} - (6,0,2)$, and $F_2 = \{\nu_3 = 0\} \cap F_1$. Concerning the sets $Z_\Delta(f_t) \cap T_{F_1}$, for 0- or 1-dimensional faces F of Δ_1, and the topological critical set of the restriction of $P_{F_1}(\mathbf{R}) \to P_{F_2}(\mathbf{R})$ to $Z_{\Delta_1}(f_t) \cap P_{F_1}(\mathbf{R})$, we can draw a picture of $\tilde{Z}_{\Delta_i}(f_t)$ in $A(\Delta_i)$, for $i = 1,2,3$. Elementary calculation shows that $Z_{\Delta_3}(f_t)$ is smooth except $t = -(\frac{7}{2})^{\frac{5}{2}}15^{\frac{1}{7}}/3 = -1.33705\cdots$. Seeing the map $P_{\Delta_3}(\mathbf{R}) \to P_{\Delta_0}(\mathbf{R})$, we can draw a picture of $\tilde{Z}_{\Delta_0}(f_t)$ in $A(\Delta_0)$. Set $I = [a,b]$, and suppose that $-(\frac{7}{2})^{\frac{5}{2}}15^{\frac{1}{7}}/3$ is not in I. As a consequence of these pictures, we obtain the followings.

1) f_t is topological trivial along I.

2) If $-(\frac{7}{2})^{\frac{5}{2}}15^{\frac{1}{7}}/3 < a < 0 < b$, then no topological trivializations of f_t along I are C^0-liftable via the (oriented) blowing up at the origin. They do not admit a strong C^0-trivialization either. The last fact is first proved by S.Koike in [8].

3) There is a C^0-liftable topological trivialization of f_t along I. (Actually there is a C^ω-liftable topological trivialization via ρ_{Δ_1}, by [5].) By Chow's lemma, this example gives a couterexample to the conjecture stated in §2 in [9].

3-2. Example comes from the Cassini's ovals. Let $C_t(x,y) = (x^2+y^2+1)^2 - 4x^2 - t$ and $I = [a,b], 1 < a < 4 < b$. The zero set of $C_t(x,y)$ is the Cassini's oval, and its picture can be find, for example, on the 48 page of "Encyclopaedia of Mathematics, Vol. 2" (Kluwer Academic Publishers, 1988). Then $(\mathbf{R}^2, C_t^{-1}(0))$ is C^0-trivial along I, but no y-level preserving C^0-trivialization along I are admitted. Let $f_t^\varepsilon(x_1,x_2,x_3) = (x_1^6 + x_1^2x_3^2 + x_2^2x_3^2)^2 - 4x_1^8x_3^2 - tx_1^4x_3^4 + x_3^8 + \varepsilon x_2^{12}$.

Set $\Delta_1 = \{(\nu_1, \nu_2, \nu_3) \in \mathbf{R}^3_+ | \nu_1 + \nu_2 + \nu_3 \geq 8, \nu_1 + \nu_2 + 2\nu_3 \geq 12\}$, $F_1 = \Delta_1 \cap \{\nu_1 + \nu_2 + 2\nu_3 = 12\}$, and $F_2 = F_1 \cap \{\nu_1 + \nu_2 + \nu_3 = 8\}$. Then F_1 majorizes F_2. The Cassini's oval is found in $Z_\Delta(f_t^0) \cap P_{F_1}$, and $P_{F_1}(\mathbf{R}) \to P_{F_2}(\mathbf{R})$ is the projection to the y-axis. Choosing ε to be a sufficiently small positive number, we have the followings.

1) f_t^ε admit MAT via ρ_{Δ_1} along I by [5].

2) No topological trivializations of f_t^ε along I are C^0-liftable via the blowing up ρ_{Δ_0}.

Since $P_{\Delta_1} \to P_{\Delta_0}$ is the blowing up, whose center is the intersection of $\rho_{\Delta_0}^{-1}(0)$ and the proper transform of $\{x_3 = 0\}$ via ρ_{Δ_0}, this also gives a counterexample to the conjecture stated in §2 in [9].

3-3. Oka's family in [11]. Using a similar analysis, we can draw a picture of Oka's family $f_t(x_1, x_2, x_3) = x_1^8 + x_2^k + x_3^k + t x_1^5 x_3^2 + x_1^3 x_2 x_3^3, (k \geq 16)$, that is neither strong nor tangent C^0-trivial near $t = 0$. (This family was first studied in [11] and, in real case, S. Koike showed this is not C^0-trivial in [8].) As a consequence of these pictures, we have that the number of connected components of the regular locus of $f^{-1}(0)$ near the origin is 4 (resp. 2), if k is even (resp. odd), etc. The detailed descriptions are left to the reader.

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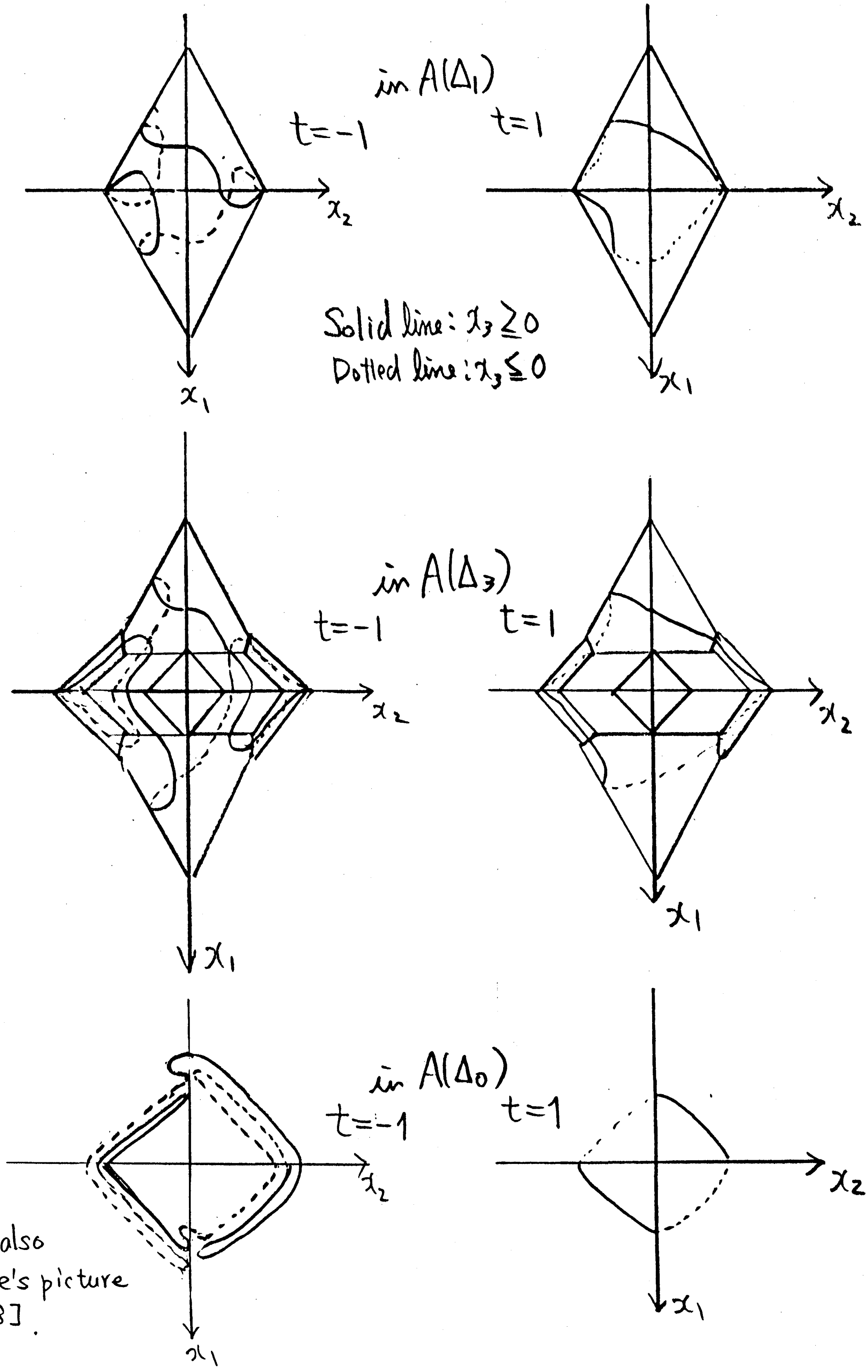
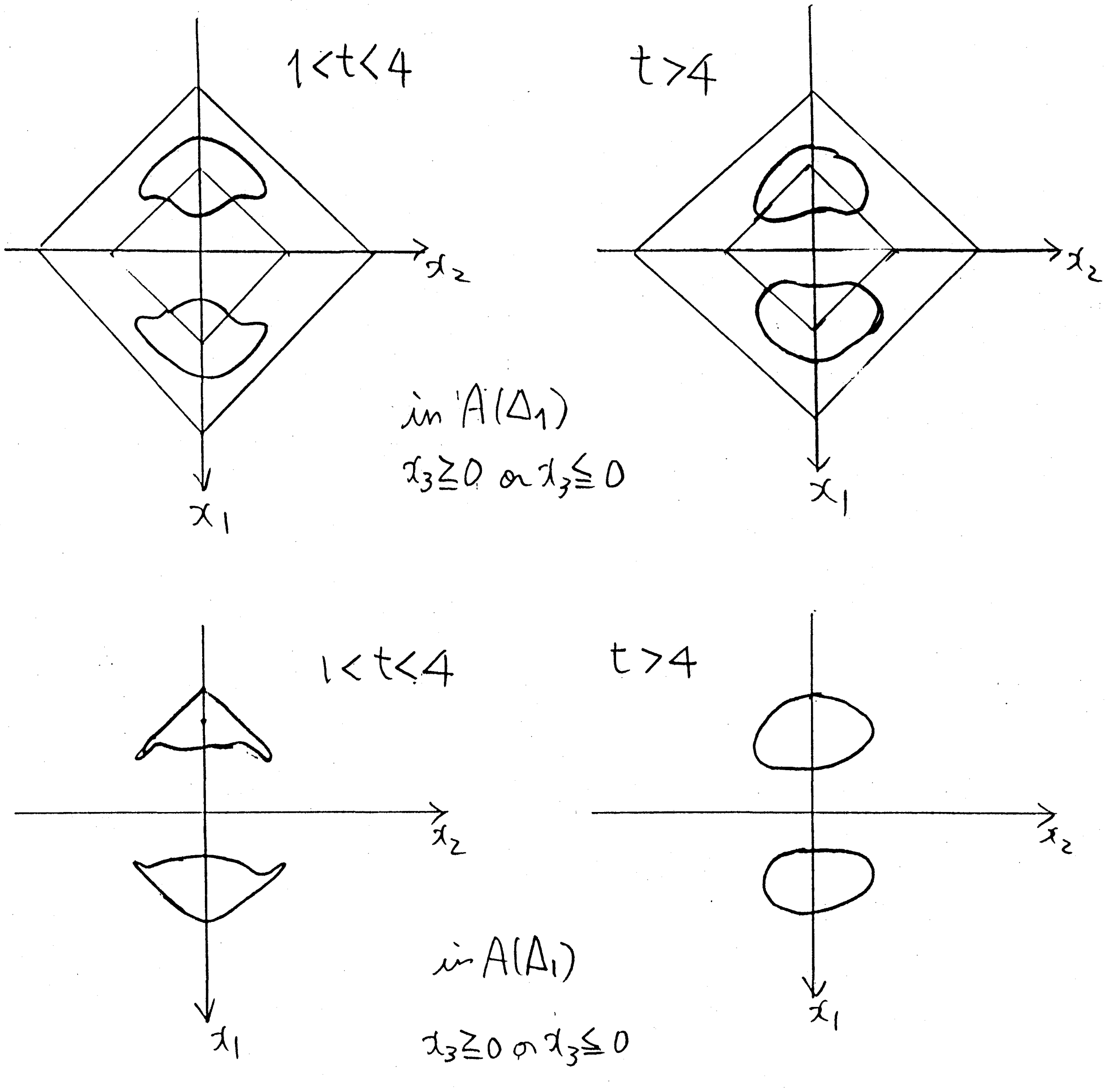
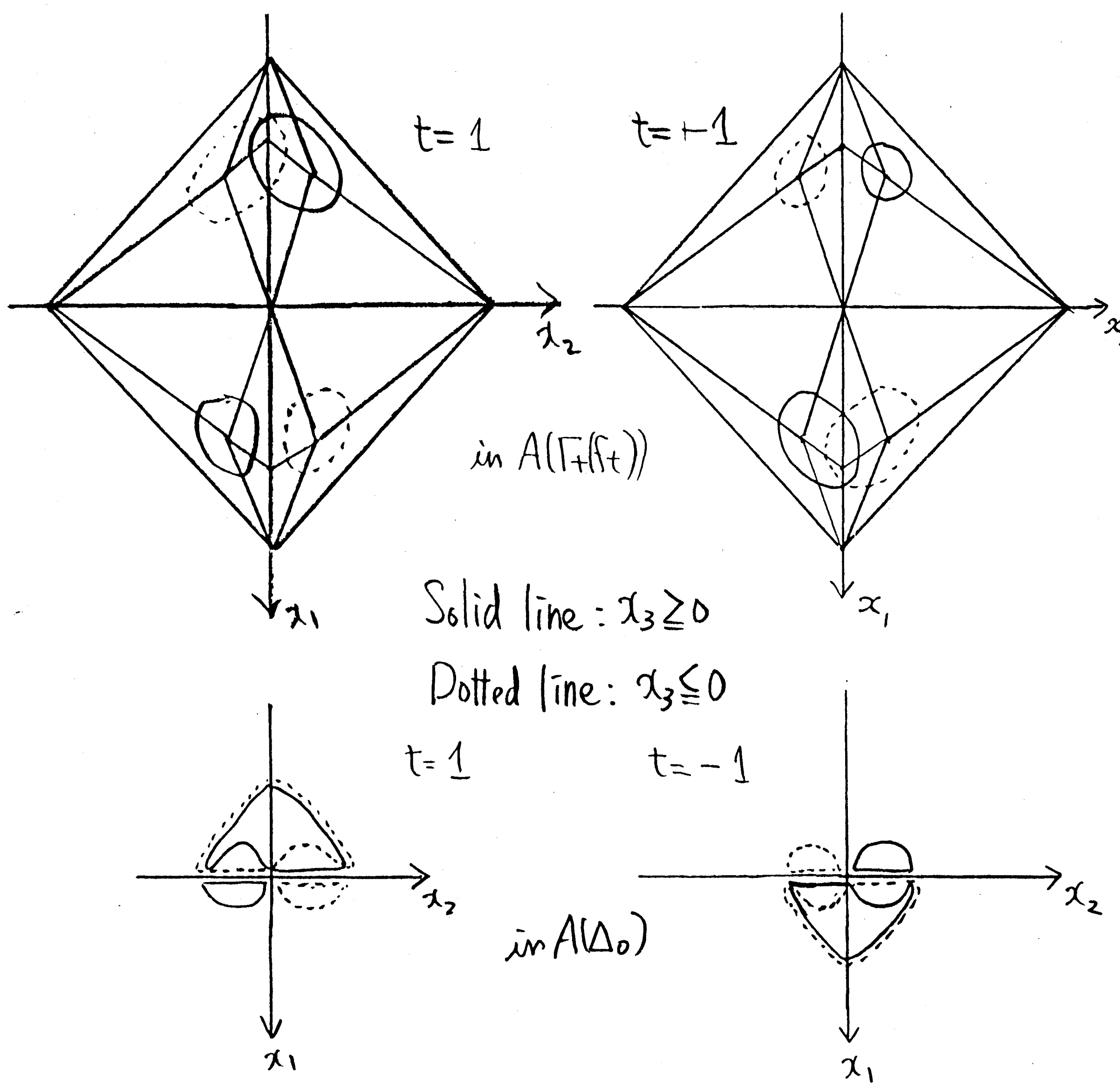


See also Koike's picture in [8].

Pictures of Brianson-Speder's family



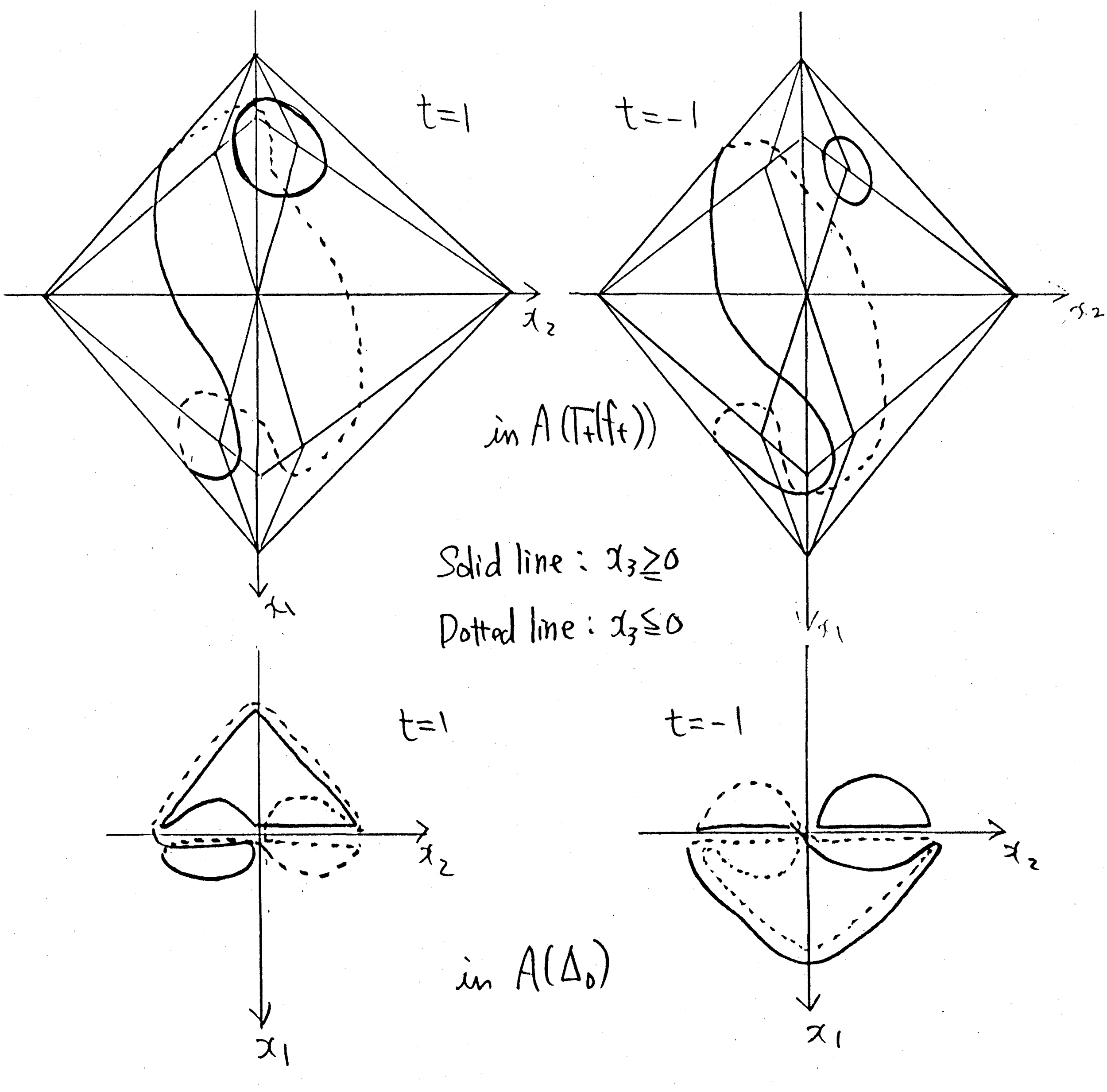
Pictures of example coming from Cassini's ovals.



Pictures of Oka's family with even k ($\geqq 18$)

The case $k=16$ is almost same.

See also Koike's pictures in [8].



Pictures of Oka's family with odd k ($\geqq 17$)